# ME 261: Numerical Analysis

## Lecture-7 & 8: Root Finding

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# Fixed Point Iternation Method for Finding Root

- Open methods **employ a formula to predict the new root from initial guess**
- The simplest formula can be developed for simple fixed-point iteration by rearranging the function f(x) = 0 so that x is on the left-hand side of the equation as:

$$x = g(x)$$

$$x^2 - 2x + 3 = 0$$

$$x = \frac{x^2 + 3}{2}$$

$$\sin x = 0$$

$$x = \sin x + x$$

$$x_{i+1} = g(x_i)$$

$$\varepsilon_a = \left| \frac{x_{i+1} - x_i}{x_{i+1}} \right| 100\%$$

Some examples of alternation form, g(x) for original function, f(x)=0

$$f(x) = x^2 - 2x + 3 = 0$$

$$\Rightarrow \quad x = \frac{x^2+3}{2} \qquad \text{in the form}, x = g_1(x)\ ; g_1(x) = \frac{x^2+3}{2}$$

$$\Rightarrow \quad x = \pm\sqrt{2x-3} \qquad \text{in the form}, x = g_2(x)\ ; g_2(x) = \pm\sqrt{2x-3}$$

$$\Rightarrow \quad x = x^2 - x + 3 \qquad \text{in the form}, x = g_3(x)\ ; g_3(x) = x^2 - x + 3$$

$$\Rightarrow \quad x = -\frac{3}{(x-2)} \qquad \text{in the form}, x = g_4(x)\ ; g_4(x) = -\frac{3}{(x-2)}$$

$$f(x) = e^{-x} - x = 0$$

$$\Rightarrow \quad x = e^{-x} \qquad \text{in the form}, x = g_1(x)\ ; g_1(x) = e^{-x}$$

$$\Rightarrow \quad x = -\ln x \qquad \text{in the form}, x = g_2(x)\ ; g_2(x) = -\ln x$$

$$f(x) = \sin x = 0$$

$$\Rightarrow \quad x = \sin x + x \qquad \text{in the form}, x = g_1(x)\ ; g_1(x) = \sin x + x$$

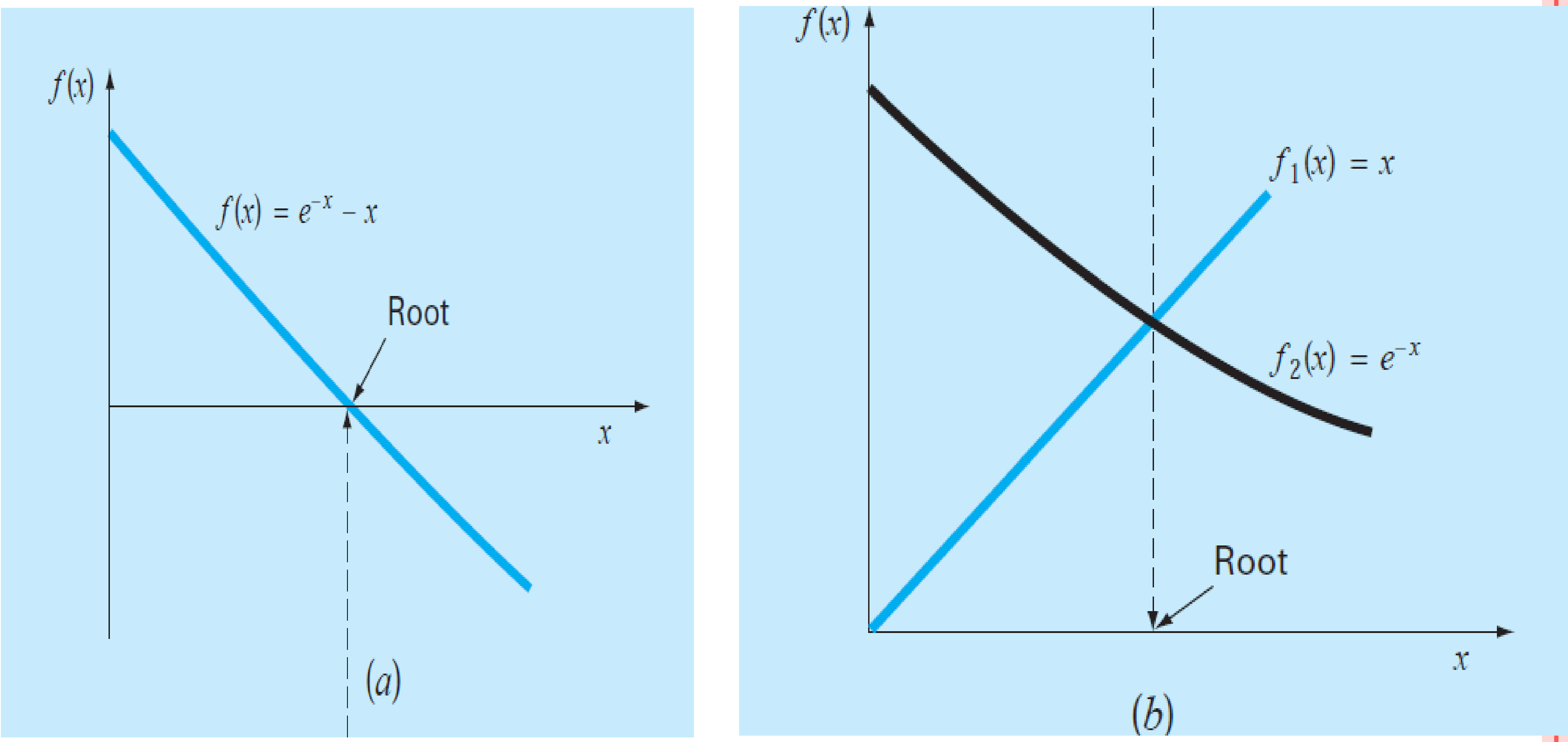
Find the root of the given equation using Fixed Point Iteration Method for an initial guess, $x_i = 0.1$. Continue root finding approximate error falls less then **0.00005%**.

$$f(x) = \frac{1.5x}{\left(1+x^2\right)^2} - 0.65\tan^{-1}\left(\frac{1}{x}\right) + \frac{0.65x}{1+x^2}$$

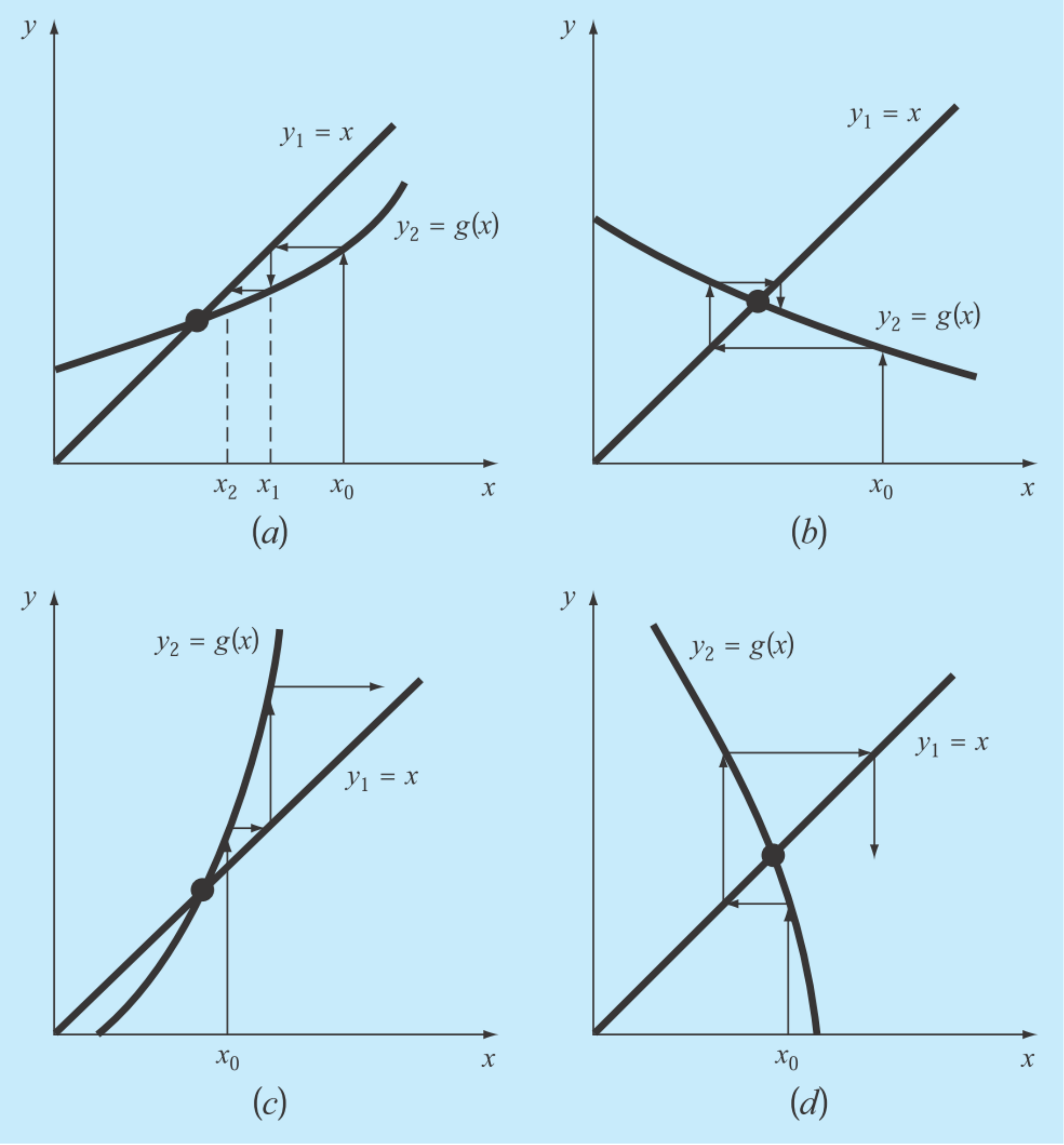
$$x = g(x) = \frac{13}{30}\left(1+x^2\right)^2 \tan^{-1}\left(\frac{1}{x}\right) - \frac{13}{30}x\left(1+x^2\right)$$

| It | xi | g(xi) | error, εr (%) |
|---|---|---|---|
| 1 | 0.1 | 0.6065355 | 83.51291925 |
| 2 | 0.6065355 | 0.4720352 | 28.49371747 |
| 3 | 0.4720352 | 0.4819150 | 2.050130461 |
| 4 | 0.481915 | 0.4807436 | 0.243664193 |
| 5 | 0.4807436 | 0.4808784 | 0.028032871 |
| 6 | 0.4808784 | 0.4808629 | 0.003237654 |
| 7 | 0.4808629 | 0.4808647 | 0.000373766 |
| 8 | 0.4808647 | 0.4808645 | 4.3151E-05 |

- Two alternative graphical methods for determining the root of $f(x) = e^{-x} - x$. *(a) Root at the* point where it crosses the *x axis; (b) root at the intersection of the component functions*.



# Graphical depiction of Converging Condition for Fixed Point Iteration Method




**(a) and (b) converging**

**(c) and (d) diverging**

Convergence occurs when

$$|g'(x)| \leq 1.0$$

$$x_{i+1} = g(x_i)$$

Suppose that the true solution is

$$x_r = g(x_r)$$

Subtracting these equations yields

$$x_r - x_{i+1} = g(x_r) - g(x_i)$$

*derivative mean-value theorem*

$$g'(\xi) = \frac{g(b) - g(a)}{b - a}$$

let $a = x_i$ and $b = x_r$,

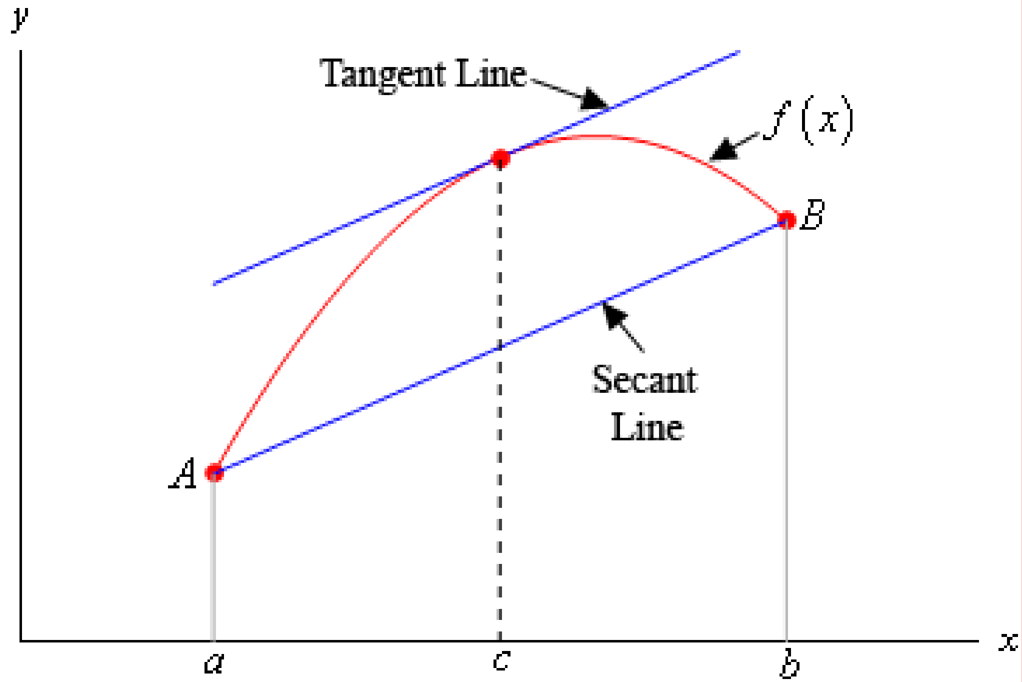
$$g(x_r) - g(x_i) = (x_r - x_i)g'(\xi)$$

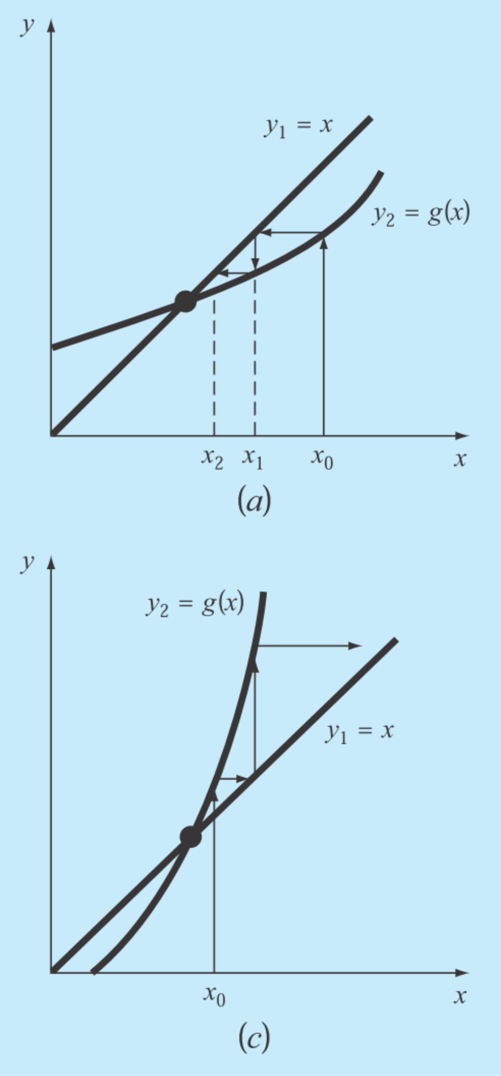
$$x_r - x_{i+1} = (x_r - x_i)g'(\xi)$$

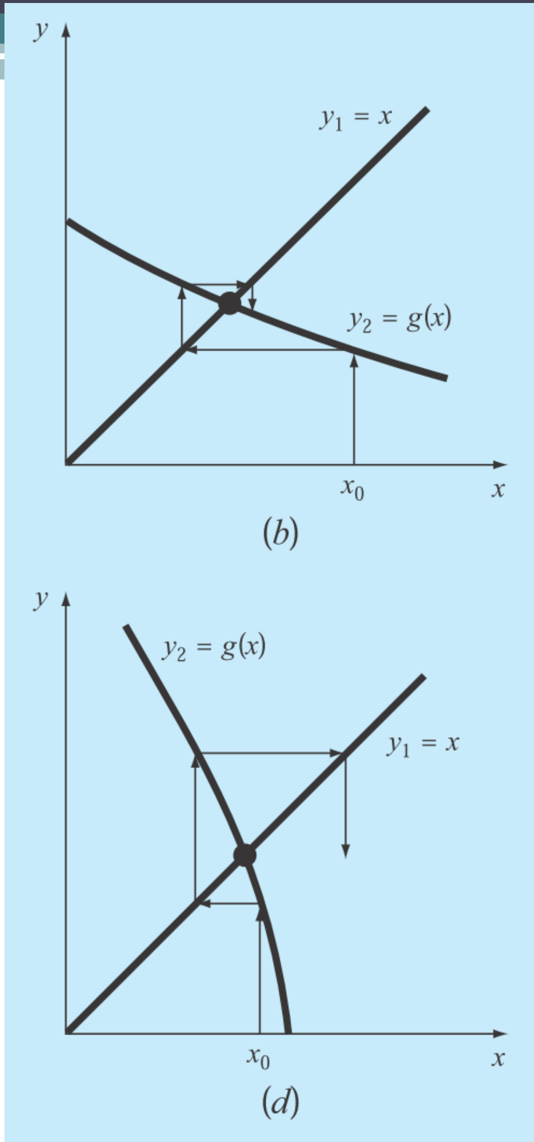
**Mean Value Theorem for Derivatives**

If $f(x)$ is continuous on [a, b] and a differentiable function over $(a,b)$, then a point c between $a$ and $b$:

$$\frac{f(b)-f(a)}{b-a} = f'(c)$$






**Monotonic propagation:**
**Root moves on one side of true root**

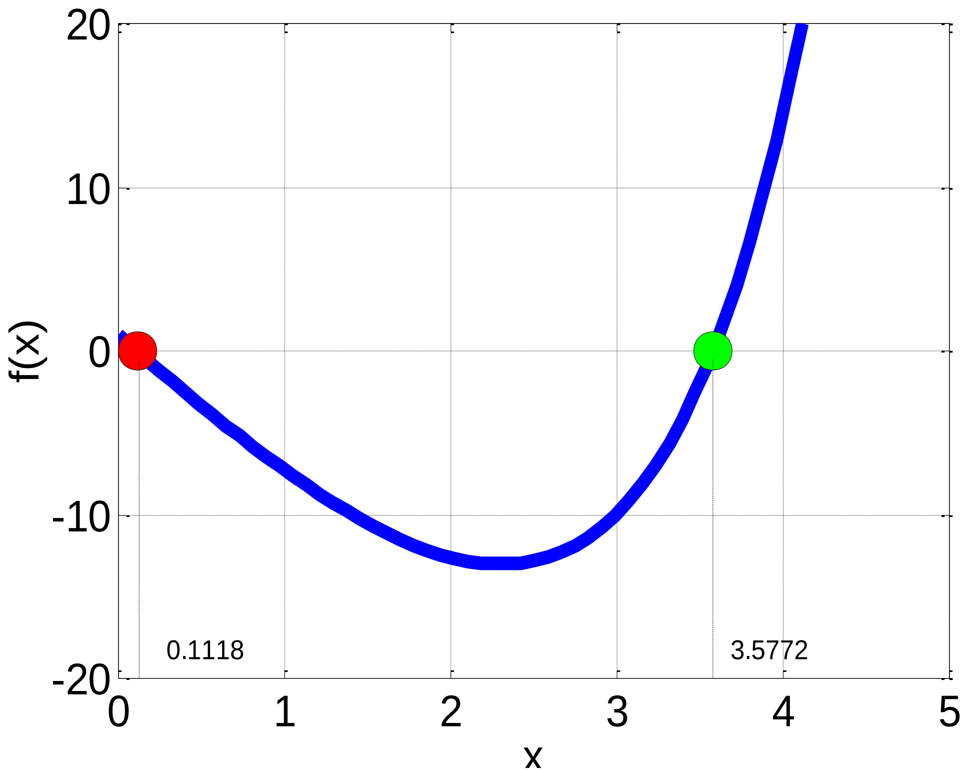
**Oscillatory propagation:**
**Root moves on both sides of true root**

# Fixed Point Iternation Method for Finding Root

Find the root for the following equation by using the fixed point iteration. Try to find the solution **with $x_0$=1.0 and $x_0$= 4.0** as your initial guesses.

$$f(x) = e^x - 10x$$

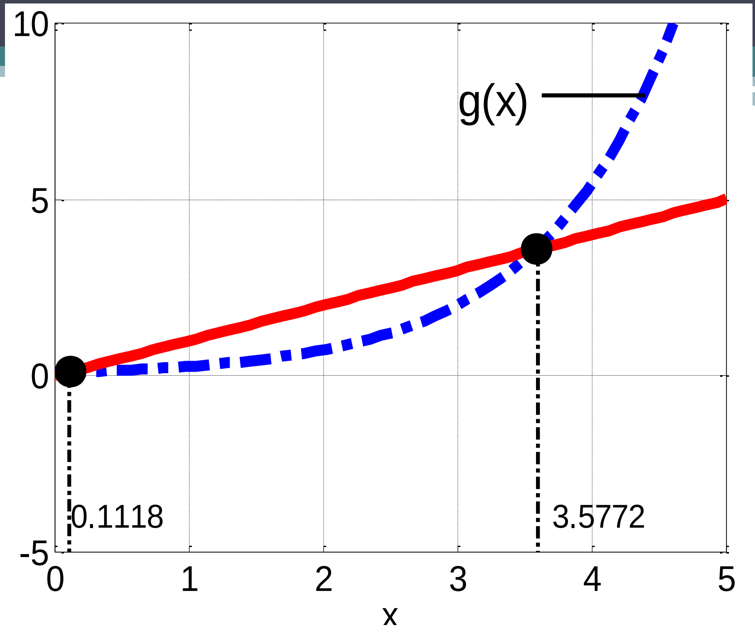
Graphical Solution




There are actually two roots to *f(x)*.
They are ***x* = 0.1118 and *x* = 3.5772**.

$$f(x) = e^x - 10x$$

$$x = g(x) = \frac{e^x}{10} \qquad x_{i+1} = \frac{e^{x_i}}{10}$$




**$x_0$ = 1.0; dg(x)/dx = 0.1 @ $x_0$**

| xi | g(xi) | Error(%) |
|---|---|---|
| 1 | 0.271828 | 267.8794 |
| 0.271828 | 0.131236 | 107.129 |
| 0.131236 | 0.114024 | 15.0955 |
| 0.114024 | 0.112078 | 1.736144 |
| 0.112078 | 0.11186 | 0.194773 |
| 0.11186 | 0.111836 | 0.02179 |
| 0.111836 | 0.111833 | 0.002437 |
| 0.111833 | 0.111833 | 0.000273 |
| 0.111833 | 0.111833 | 3.05E-05 |

**$x_0$ = 4.0; dg(x)/dx = 5.4598 @ $x_0$**

| xi | g(xi) | Error(%) |
|---|---|---|
| 4 | 5.459815 | 26.73744 |
| 5.459815 | 23.50539 | 76.77208 |
| 23.50539 | 1.62E+09 | 100 |
| 1.62E+09 | #NUM! | #NUM! |
| #NUM! | #NUM! | #NUM! |
| #NUM! | #NUM! | #NUM! |
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- If we start off the solution with $x_0$=1.0, we will converge to the root at $x$ = 0.1118 since $|dg/dx(x_0)| < 1.0$.
- If we start the solution with $x_0$=4.0, will NOT converge to the root at $x$=3.5772 and the solution will blow up since $|dg/dx(x_0)| > 1.0$!!

*Exercise*: Let $f(x) = x^2 - 2x - 3 = 0$

a) Find the two roots for *f(x)* analytically.
b) Show that *f(x)* can be rearranged into the following forms called transposition:

$$x = g_1(x) = \sqrt{2x+3}$$

$$x = g_2(x) = \frac{3}{x-2}$$

$$x = g_3(x) = \frac{(x^2-3)}{2}$$

c) With these three expressions for $g_i(x)$, use the fixed point iteration method to find the approximate root for $f(x)$. Start with the initial guess of $\mathbf{x_o=4}$.

Compare your solutions to the answer you get in part a).

d) Sketch all the $g_i(x)$ and show what is happening graphically

$x_0 = 4$

| Iteration | $x = g_1(x) = \sqrt{2x+3}$ | | $x = g_2(x) = \frac{3}{(x-2)}$ | | $x = g_3(x) = \frac{x^2-3}{2}$ | |
|---|---|---|---|---|---|---|
| | **x** | $\varepsilon_a$ | **x** | $\varepsilon_a$ | **x** | $\varepsilon_a$ |
| 1 | 3.31662 | - | 1.5 | - | 6.5 | - |
| 2 | 3.10375 | 6.85% | -6 | 125% | 19.625 | 66.8% |
| 3 | 3.03439 | 2.29% | -0.375 | 1500% | 191.07 | 89.7% |
| 4 | 3.01144 | 0.76% | -1.26316 | 236.8% | 18252.3 | 98.9% |
| 5 | 3.00381 | 0.25% | -0.91935 | 37.4% | 166573226.1 | 99.9% |
| 6 | 3.00127 | 0.08% | -1.02763 | 10.5% | | |
| 7 | 3.00042 | 0.03% | -0.99087 | 3.7% | | |
| 8 | 3.00014 | 0.009% | -1.00305 | 1.2% | | |
| 9 | 3.00005 | 0.003% | -0.99898 | 0.4% | | |
| 10 | 3.00002 | 0.001% | -1.00034 | 0.1% | | |

**Converging towards root x=3**

**Converging towards other root x=-1**

**Diverging from the root**

Given function, $x^2 - 2x = 3$

$$\therefore\ f(x) = x^2 - 2x - 3 = 0$$

$x_0 = 4$

3 alternate forms, g(x) for the solution of function, f(x) = 0

$$x = g_1(x) = \sqrt{2x+3} \qquad \textbf{(A)}$$

$$x = g_2(x) = \frac{3}{(x-2)} \qquad \textbf{(B)}$$

$$x = g_3(x) = \frac{x^2-3}{2} \qquad \textbf{(C)}$$

**Convergence behavior depends on the nature of the alternate form, g(x)**

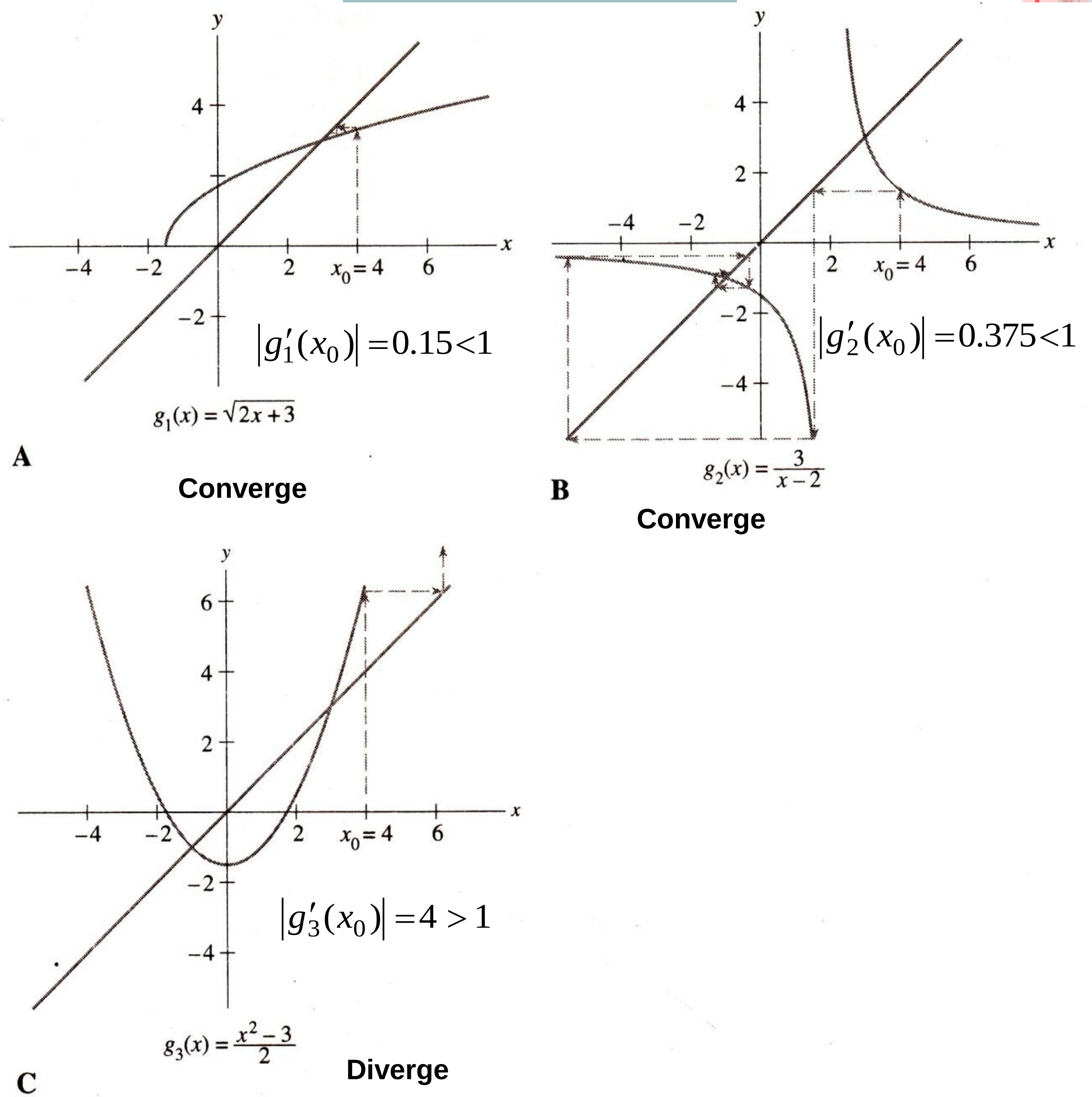



**Fig. Iterative progress of Fixed point Iteration method by graphical approach**

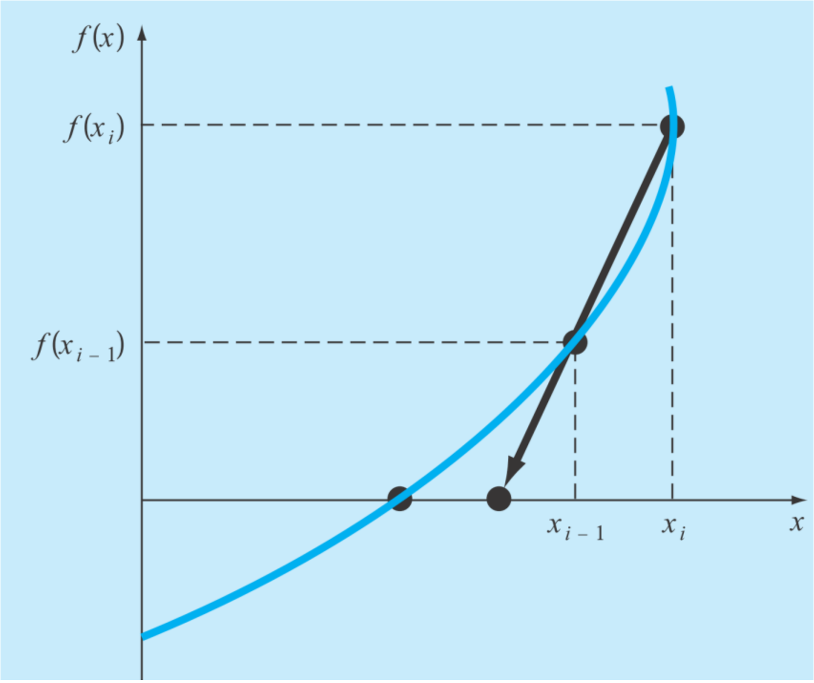
Note that:

- It does not mean that if you have $|dg(x)/dx \ @ \ x_0| > 1$, the solution process will blow up.
- Use this condition only as a good guide.
- Sometimes you can still get a converged solution if you start have $|dg(x)/dx \ @ \ x_0| > 1$. It really depends on the problem to solve.

# Secant Method for Finding Root

Sometimes the derivative of a function **is difficult or inconvenient to evaluate** which is a requirement of Newton's method. For these cases, the **derivative can be approximated by a backward finite difference approach**, as in-

$$f'(x_i) \approx \frac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i}$$




**Newton's Formula:**

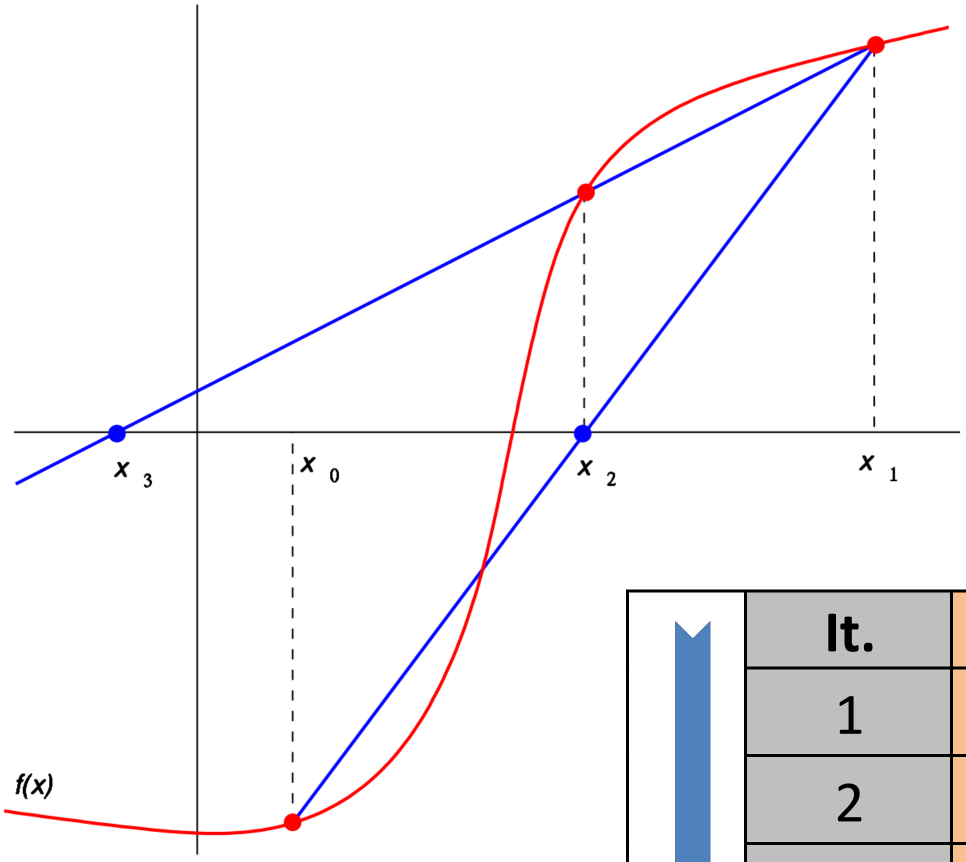
$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$\Rightarrow x_{i+1} = x_i - \frac{f(x_i)}{\dfrac{f(x_{i-1}) - f(x_i)}{x_{i-1} - x_i}}$$

$$\boxed{\begin{array}{l} \textit{Secant Formula:} \\ x_{i+1} = x_i - \dfrac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)} \end{array}}$$

Secant method requires **two initial guesses** of x. **(But keep in mind that the value of f(x) is not required to change sign between the guesses)**

# Graphical Interpretation of Secant Method

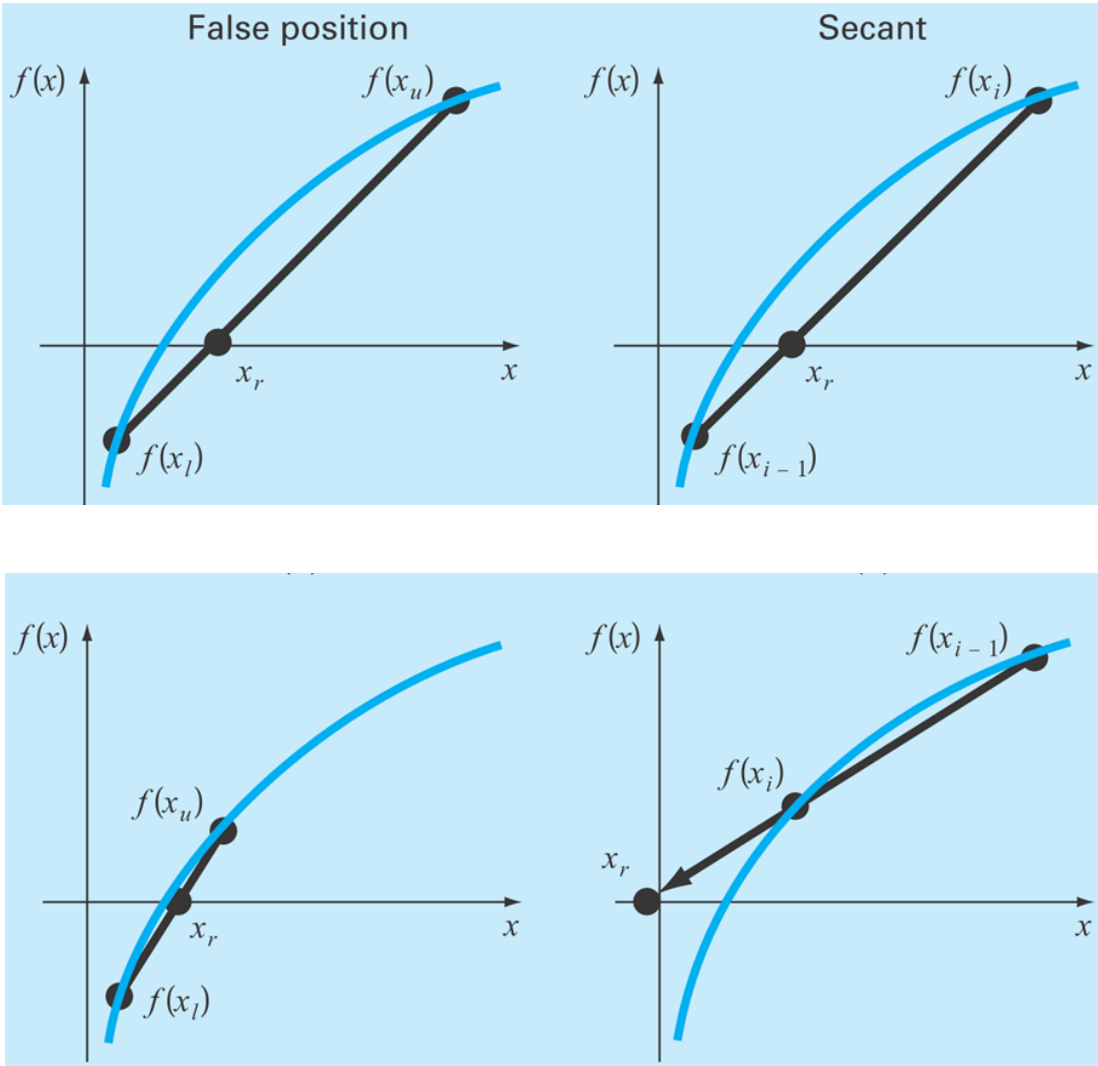



*Secant Formula:*

$$x_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$$

| It. | Known | | New |
|---|---|---|---|
| 1 | $x_0$ | $x_1$ | $x_2$ |
| 2 | $x_1$ | $x_2$ | $x_3$ |
| 3 | $x_2$ | $x_3$ | $x_4$ |
| 4 | $x_3$ | $x_4$ | $x_5$ |

# False Position Method v.s Secant Method



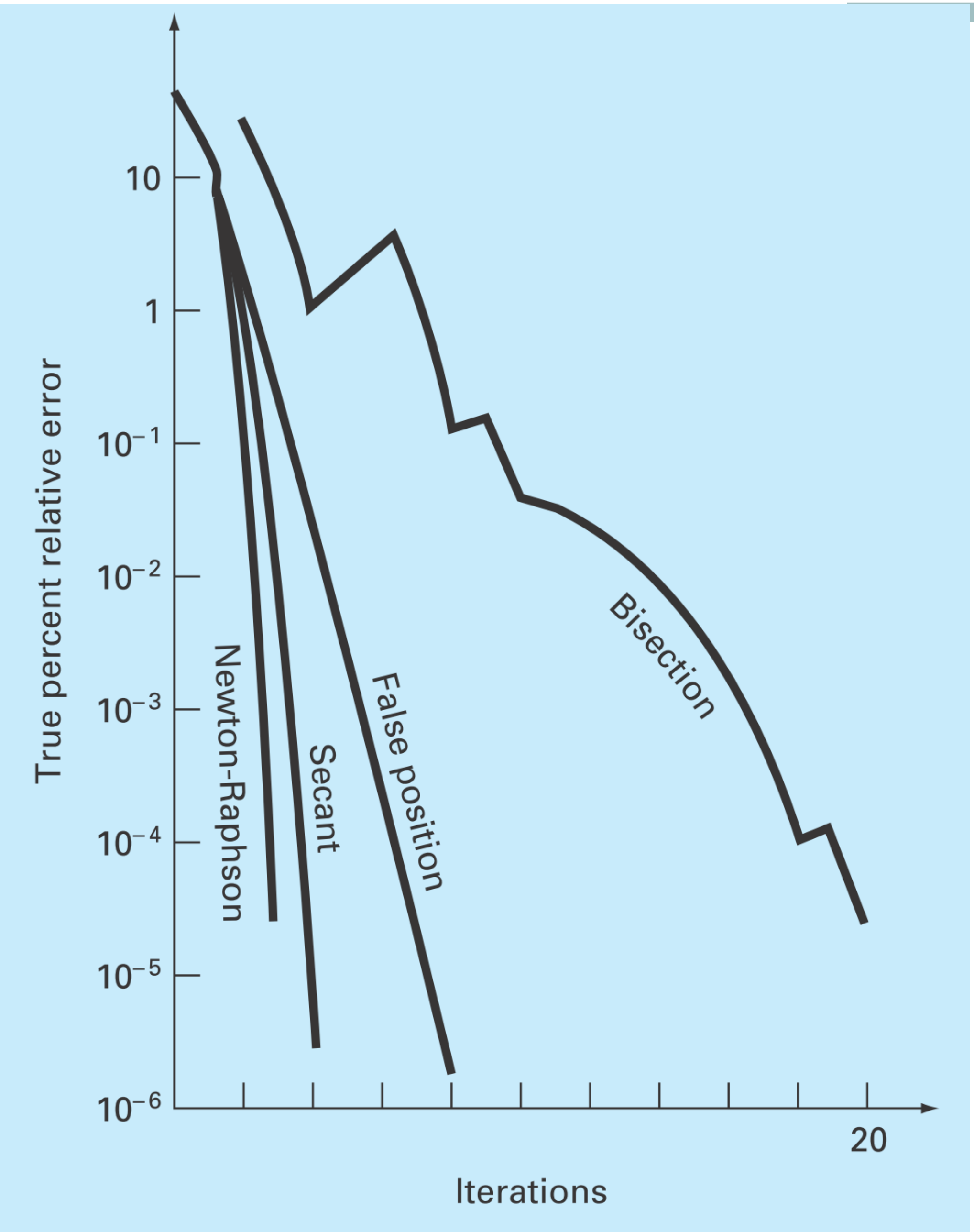

**Current Step**

**Next Step**

**Find a root of the equation $x^3 - 8x - 5 = 0$ using the secant method up to six significant digit.**

$$\text{Secant Formula:}$$
$$x_{i+1} = x_i - \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$$

| It | $x_0$ | $f(x_0)$ | $x_1$ | $f(x_1)$ | $x_2$ | $f(x_2)$ | Error (%) |
|---|---|---|---|---|---|---|---|
| 1 | 3 | -2 | 3.5 | 9.875 | 3.084211 | -0.33558 | 13.48123 |
| 2 | 3.5 | 9.875 | 3.084211 | -0.33558 | 3.097876 | -0.0532 | 0.441119 |
| 3 | 3.0842105 | -0.33558 | 3.097876 | -0.0532 | 3.100451 | 0.000388 | 0.083045 |
| 4 | 3.0978758 | -0.0532 | 3.100451 | 0.000388 | 3.100432 | -4.4E-07 | 0.000601 |
| 5 | 3.1004506 | 0.000388 | 3.100432 | -4.4E-07 | 3.100432 | -3.7E-12 | 6.86E-07 |




Comparison of the true percent relative errors $\varepsilon_t$ *for the methods to determine the roots of* $f(x) = e^{-x} - x$.

**Fig.: Characteristics of Error propagation in various root finding techniques**